

MTH250 - CALCULUS

Lecture No. 24

DR. ABDUL WAHAB

Assistant Professor,
Department of Mathematics, CIIT, Pakistan.

© **VIRTUAL CAMPUS**

COMSATS Institute of Information Technology
CIIT, Virtual Campus Secretariat 402, Street 34, I-8/2 Islamabad-Pakistan.
Ph:+92 51 9101237-39 | info@vcomsats.edu.pk | vcomsats.edu.pk

These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

Contents

Contents	i
24 Multivariable functions II	1
24.1 Tangent plane	1
24.2 Differentiability of multivariable functions	2
24.3 Differentials and linear approximations	3
24.4 Chain rules for partial differentiation	6
24.4.1 Implicit differentiation	7
24.5 Extrema of multivariable functions	8

LECTURE 24

Multivariable functions II

24.1 Tangent plane

Definition 24.1 (Tangent plane). Let $f = f(x, y)$ be a function of two variables and let $P(x_0, y_0, f(x_0, y_0))$ be a point on the graph of f . Suppose C_1 and C_2 are the intersections of the surface $z = f(x, y)$ with the plane $y = y_0$ and $x = x_0$ respectively. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at point (x_0, y_0) . Then, if it exists, the plane containing T_1 and T_2 is called the tangent plane to f at point (x_0, y_0) .

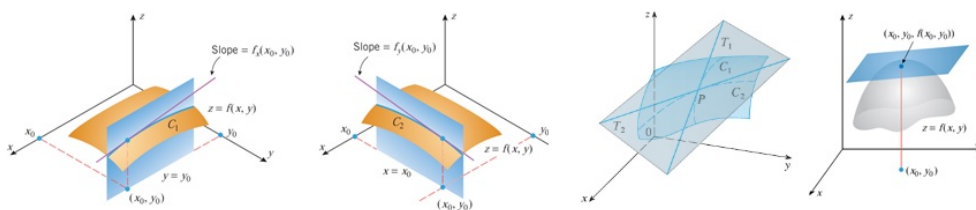


Figure 24.1: Ideas of a tangent plane.

Recall that the equation of the plane in point-normal form passing through point $P(x_0, y_0, z_0)$ is of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Assuming that the plane is not vertical, we have $C \neq 0$. Therefore, the equation of the plane can be rewritten as

$$z - z_0 = a(x - x_0) + b(y - y_0).$$

The tangent line to C_1 at P is obtained by taking $y = y_0$ i.e.

$$z - z_0 = a(x - x_0).$$

Since $f_x(x_0, y_0)$ is the slope of the tangent line to C_1 at P . Therefore, $a = f_x(x_0, y_0)$. Similarly, $b = f_y(x_0, y_0)$. Thus the equation of the tangent plane is

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 24.1. Find the equation of the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution. Let $f(x, y) = 2x^2 + y^2$. Then $f_x(x, y) = 4x$ and $f_y(x, y) = 2y$ so that $f_x(1, 1) = 4$ and $f_y(1, 1) = 2$. Hence, the equation of the tangent plane at $(1, 1, 3)$ is given by

$$z = 3 + 4(x - 1) + 2(y - 1) \quad \text{or} \quad z = 4x + 2y - 3.$$

□

24.2 Differentiability of multivariable functions

Definition 24.2. A function f of two variables is said to be differentiable at (x_0, y_0) provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

A function f of three variables is said to be differentiable at (x_0, y_0, z_0) provided $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x, \Delta y, \Delta z) \rightarrow (0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x - f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0$$

Example 24.2. Prove that $f(x, y) = x^2 + y^2$ is differentiable at $(0, 0)$.

Solution. The increment is

$$\Delta f = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = (\Delta x)^2 + (\Delta y)^2$$

Since $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, we have $f_x(0, 0) = f_y(0, 0) = 0$, and

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta x)^2 + (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \sqrt{(\Delta x)^2 + (\Delta y)^2} = 0$$

Therefore, f is differentiable at $(0, 0)$.

□

Remark 24.3. Let us define for $(\Delta x, \Delta y) \neq (0, 0)$

$$\epsilon := \epsilon(\Delta x, \Delta y) = \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}},$$

and for $\epsilon(0, 0) := 0$. By definition of differentiability we have

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \epsilon(\Delta x, \Delta y) = 0.$$

Immediately, by definition of ϵ , we have

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon\sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

In other words, if f is differentiable at (x_0, y_0) , then Δf may be expressed as shown above, where $\epsilon \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$ and further $\epsilon = 0$ when $(\Delta x, \Delta y) = (0, 0)$.

Theorem 24.4. If a function is differentiable at a point, then it is continuous at that point.

Proof. Assume that f is differentiable at (x_0, y_0) . To prove that f is continuous at (x_0, y_0) we must show that

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0)$$

By this $\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \Delta f = 0$. However,

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \Delta f = \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \left[f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2} \right] = 0$$

□

Theorem 24.5. If all first-order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.

Example 24.3. Consider the function $f(x, y, z) = x + yz$. Since

$$f_x(x, y, z) = 1, \quad f_y(x, y, z) = z, \quad f_z(x, y, z) = y$$

are defined and continuous everywhere. We conclude that f is differentiable everywhere.

24.3 Differentials and linear approximations

Definition 24.6 (Total differential).

If $z = f(x, y)$ is differentiable at a point (x_0, y_0) , we let

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

denote a new function with dependent variable dz and independent variables dx and dy . We refer to this function (also denoted df) as the total differential of z at (x_0, y_0) or as the total differential of f at (x_0, y_0) . Similarly, for a function $w = f(x, y, z)$ of three variables we have the total differential of w at (x_0, y_0, z_0) ,

$$dw = f_x(x_0, y_0, z_0)dx + f_y(x_0, y_0, z_0)dy + f_z(x_0, y_0, z_0)dz$$

which is also referred to as the total differential of f at (x_0, y_0, z_0) . It is common practice to omit the subscripts and write as

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

As with the one-variable case, the approximations

$$\Delta f \simeq f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

for a function of two variables and the approximation

$$\Delta f \simeq f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z$$

for a function of three variables have a convenient formulation in the language of differentials. In the two variable case, the approximation

$$\Delta f \simeq f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

can be written in the form $\Delta f \simeq df$ for $dx = \Delta x$ and $dy = \Delta y$.

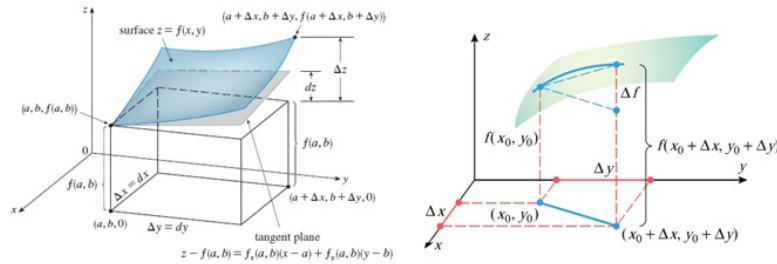


Figure 24.2: Differentials and linear approximations of multi-variable functions.

Example 24.4. Approximate the change in $z = xy^2$ from its value at $(0.5, 1)$ to its value at $(0.503, 1.004)$. Compare the magnitude of the error in this approximation with the distance between the points $(0.5, 1)$ and $(0.503, 1.004)$.

Solution. For $z = xy^2$ we have $dz = y^2 dx + 2xy dy$. Evaluating this differential at $(x, y) = (0.5, 1.0)$, using

$$dx = \Delta x = 0.503 - 0.5 = 0.003 \quad \text{and} \quad dy = \Delta y = 1.004 - 1.0 = 0.004$$

we have

$$dz = 1.0^2(0.003) + 2(0.5)(1.0)(0.004) = 0.007.$$

Since, $z = 0.5$ at $(x, y) = (0.5, 1.0)$ and $z = 0.507032048$ at $(x, y) = (0.503, 1.004)$, we have

$$\Delta z = 0.507032048 - 0.5 = 0.007032048$$

and the error in approximating Δz by dz has magnitude

$$|dz - \Delta z| = |0.007 - 0.007032048| = 0.000032048.$$

Since the distance between $(0.5, 1.0)$ and $(0.503, 1.004) = (0.5 + \Delta x, 1.0 + \Delta y)$ is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(0.003)^2 + (0.004)^2} = \sqrt{0.000025} = 0.005,$$

we have

$$\frac{|dz - \Delta z|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{0.000032048}{0.005} = 0.0064096 < \frac{1}{150}.$$

Thus, the magnitude of the error in our approximation is less than $\frac{1}{150}$ of the distance between the two points. $\square$

Definition 24.7 (Local linear approximation).

Since the tangent plane to the surface $z = f(x, y)$ at $P(x_0, y_0, f(x_0, y_0))$ is very close to the surface at least when it is near P , we may use the function defining the tangent plane as a linear approximation to f . Recall that the equation of the tangent plane to the graph of $f(x, y)$ at P is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

When $f(x, y)$ is differentiable at (x_0, y_0) we refer to

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

as the local linear approximation to f at (x_0, y_0) .

For a function $f(x, y, z)$ that is differentiable at (x_0, y_0, z_0) the local linear approximation is

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0).$$

Example 24.5. Let $L(x, y)$ denote the local linear approximation to $f(x, y) = \sqrt{x^2 + y^2}$ at the point $(3, 4)$. Compare the error in approximating

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}$$

by $L(3.04, 3.98)$ with the distance between the point $(3, 4)$ and $(3.04, 3.98)$.

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

with $f_x(3, 4) = \frac{3}{5}$ and $f_y(3, 4) = \frac{4}{5}$. Therefore, the local linear approximation to f at $(3, 4)$ is given by

$$L(x, y) = 5 + \frac{3}{5}(x - 3) + \frac{4}{5}(y - 4)$$

Consequently,

$$f(3.04, 3.98) \approx L(3.04, 3.98) = 5 + \frac{3}{5}(0.04) + \frac{4}{5}(y - 4) = 5.008$$

Since

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819$$

the error in the approximation is about $5.00819 - 5.008 = 0.00019$. This is less than $\frac{1}{200}$ of the distance

$$\sqrt{(3.04 - 3)^2 + (3.98 - 4)^2} \approx 0.045$$

between the points $(3, 4)$ and $(3.04, 3.98)$.

24.4 Chain rules for partial differentiation

Definition 24.8 (Chain rules).

If $x = x(t)$ and $y = y(t)$ are differentiable at t and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y) .

If $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v) and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(u, v), y(u, v))$, then

$$z = f(x(u, v), y(u, v))$$

has first order partial derivatives at the point (u, v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}.$$

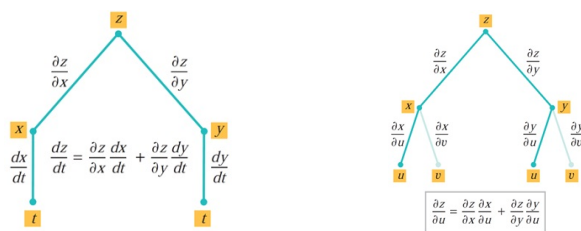


Figure 24.3: Chain rule: For differentiating $z = f(x(t), y(t))$, with respect to x and y (left). For partial derivatives (right)

Example 24.6. Suppose that $z = x^2y$, $x = t^2$, and $y = t^3$. Then by chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (2xy)(2t) + (x^2)(3t^2) = (2t^5)(2t) + (t^4)(3t^2) = 7t^6.$$

Example 24.7. Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$

Find $\frac{\partial w}{\partial \rho}$ and $\frac{\partial w}{\partial \theta}$.

Solution. From the chain rule, we obtain,

$$\begin{aligned} \frac{\partial w}{\partial \rho} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial \rho} \\ &= 2x \sin \phi \cos \theta + 2y \sin \phi \sin \theta - 2z \cos \phi \\ &= 2\rho \sin^2 \phi \cos^2 \theta + 2\rho \sin^2 \phi \sin^2 \theta - 2\rho \cos^2 \phi \\ &= 2\rho \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) - 2\rho \cos^2 \phi \\ &= 2\rho (\sin^2 \phi - \cos^2 \phi) = -2\rho \cos 2\phi, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial \theta} \\ &= 2x(-\rho \sin \phi \sin \theta) + 2y\rho \sin \phi \cos \theta - 0 \\ &= -2\rho^2 \sin^2 \phi \sin \theta \cos \theta + 2\rho^2 \sin^2 \phi \sin \theta \cos \theta \\ &= 0 \end{aligned}$$

□

24.4.1 Implicit differentiation

Remark 24.9. Let $z = f(x, y)$ be a differentiable function of x . Then

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}.$$

When $f(x, y) = \text{constant}$, then

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0$$

Theorem 24.10. If the equation $f(x, y) = \text{constant}$ defines y implicitly as a differentiable function of x , and if $\frac{\partial f}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}.$$

Example 24.8. Given that $x^3 + y^2x - 3 = 0$, find $\frac{dy}{dx}$.

Solution. Let

$$f(x, y) = x^3 + y^2x - 3.$$

Then

$$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{3x^2 + y^2}{2xy}.$$

□

24.5 Extrema of multivariable functions

Definition 24.11 (Extrema of multivariable functions).

A function f of two variables is said to have a **relative maximum** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an **absolute maximum** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f .

A function f of two variables is said to have a **relative minimum** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an **absolute minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f .

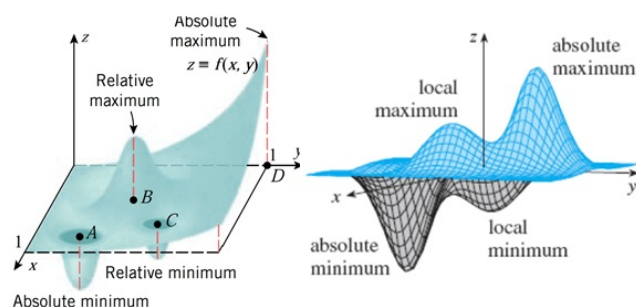


Figure 24.4: Extrema of multivariable functions.

Theorem 24.12 (Extremum value). *If $f(x, y)$ is continuous on a closed and bounded set R , then f has both an absolute maximum and an absolute minimum on R .*

Definition 24.13. *A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a critical point of the function if*

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$$

or if at least one partial derivatives do not exist at (x_0, y_0) .

Definition 24.14 (Saddle point).

If $f(x, y)$ has neither a relative maximum nor a relative minimum at a critical point (x_0, y_0) , then it is called a saddle point of f . In general, the surface $z = f(x, y)$ has a saddle point at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum and the trace in the other has a relative minimum at (x_0, y_0) .

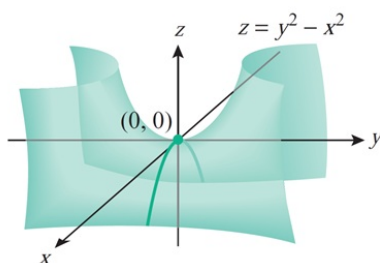


Figure 24.5: Saddle point: The function $f(x, y) = y^2 - x^2$ has neither a relative maximum nor a relative minimum at the critical point $(0, 0)$.

Theorem 24.15 (The second partials test).

Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0) , and let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

1. *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0) .*
2. *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0) .*
3. *If $D < 0$, then f has a saddle point at (x_0, y_0) .*
4. *If $D = 0$, then no conclusion can be drawn.*

Example 24.9. *Locate all relative extrema and saddle points of*

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y.$$

Solution. Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y$, the critical points of f satisfy the equations

$$6x - 2y = 0 \quad \text{and} \quad -2x + 2y - 8 = 0.$$

Solving these equations for x and y yields $x = 2$ and $y = 6$, so $(2, 6)$ is the only critical point. To apply the second partials test, we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2.$$

At the point $(2, 6)$, we have

$$D = f_{xx}(2, 6)f_{yy}(2, 6) - f_{xy}^2(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and $f_{xx}(2, 6) = 6 > 0$ so f has a relative minimum at $(2, 6)$. □